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DESIGN OF CIRCULAR CYLINDRICAL SHELLS OF MINIMUM WEIGHT WITH FIXED NATURAL OSCILLATION FREQUENCIES*

A.S. BRATUS'

Approximate solutions are obtained, using asymptotic methods, of the problem of the optimum design of cylindrical shells of variable thickness, of minimum weight for fixed natural oscillation frequencies in the axis-ymmetric and non-axisymmetric cases. Qualitative patterns of the thickness distribution for optimum solutions are obtained and analyzed.

1. Basic equations.

Consider the natural oscillations of a circular cylindrical shell of variable thickness. We assume that the mean surface is specified in curvilinear coordinates x and α in such a way that the first quadratic form has the form $R^2(dx^2+d\alpha^2)$, where R is the radius of the circular cylindrical shell, x varies along the generatrix, and α is an angular coordinate that varies in the transverse direction. We shall consider shells with straight cutoffs, that, in dimensionless variables (x, α), occupy the rectangular region.

$$D=\{x, \alpha: 0 \leqslant x \leqslant k,\ 0 \leqslant \alpha \leqslant \alpha_0 < 2\pi\},\ k=l/R$$

where l is the shell length.

The set of equations in displacements, which determines the natural oscillations of a circular cylindrical shell of variable thickness $h(x,\alpha)$ can be expressed (e.g., /1/) in the form

$$A(h)z(x,\alpha)=\lambda hz(x,\alpha);\ A(h)=\|A_{ij}(h)\|_{i,j=1,2,3} \tag{1.1}$$

$$\lambda=\rho\frac{R^2(1-\mu^2)}{E}\omega^2,\quad A_{11}=-\frac{\partial}{\partial x}h\frac{\partial}{\partial x}-\frac{1-\mu}{2}\frac{\partial}{\partial\alpha}h\frac{\partial}{\partial\alpha}$$

$$A_{12}=-\mu\frac{\partial}{\partial x}h\frac{\partial}{\partial\alpha}-\frac{1-\mu}{2}\frac{\partial}{\partial\alpha}h\frac{\partial}{\partial x}$$

$$A_{13}(h)=\mu\frac{\partial}{\partial x}h,\quad A_{31}(h)=-\mu h\frac{\partial}{\partial x}$$

$$A_{21}(h)=-\frac{(1-\mu)}{2}\frac{\partial}{\partial x}h\frac{\partial}{\partial\alpha}-\mu\frac{\partial}{\partial\alpha}h\frac{\partial}{\partial x}$$

$$A_{22}(h)=-\frac{(1-\mu)}{2}\frac{\partial}{\partial x}h\frac{\partial}{\partial x}-\frac{\partial}{\partial\alpha}h\frac{\partial}{\partial\alpha}-$$
$$\delta_0^2\left[2(1-\mu)\frac{\partial}{\partial x}h^3\frac{\partial}{\partial x}+\frac{\partial}{\partial\alpha}h^3\frac{\partial}{\partial\alpha}\right]$$

$$A_{23}(h)=\frac{\partial}{\partial\alpha}h-\delta_0^2\left[2(1-\mu)\frac{\partial}{\partial x}h^3\frac{\partial^2}{\partial x\,\partial\alpha}+\right.$$
$$\left.\mu\frac{\partial}{\partial\alpha}h^3\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial\alpha}h^3\frac{\partial^2}{\partial\alpha^2}\right]$$

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$$A_{32}(h) = -h\frac{\partial}{\partial\alpha} + \delta_0^2\left[2(1-\mu)\frac{\partial^2}{\partial x\partial\alpha}h^3\frac{\partial}{\partial x} + \mu\frac{\partial^2}{\partial x^2}h^3\frac{\partial}{\partial\alpha} + \frac{\partial^2}{\partial\alpha^2}h^3\frac{\partial}{\partial\alpha}\right]$$

$$A_{33}(h) = h + \delta_0^2\left[\frac{\partial^2}{\partial x^2}h^3\left(\frac{\partial^2}{\partial x^2} + \mu\frac{\partial^2}{\partial\alpha^2}\right) + \frac{\partial^2}{\partial\alpha^2}h^3\left(\frac{\partial^2}{\partial\alpha^2} + \mu\frac{\partial^2}{\partial x^2}\right) + 2(1-\mu)\frac{\partial^2}{\partial x\partial\alpha}h^3\frac{\partial^2}{\partial x\partial\alpha}\right]$$

where μ is Poisson's ratio, $\delta_0^2 = (12R^2)^{-1}$, and $h = h(x, \alpha)$ is the shell thickness.

We denote by $z(x, \alpha)$ in (1.1) the vector function of displacements $z^* = (u(x, \alpha), v(x, \alpha) w(x, \alpha))$, where u, v, w are the displacement components in the directions of the generatrix, the directrix, and of the normal to the cylindrical shell, respectively, λ is the problem eigenvalue, ρ is the material density, E is Young's modulus, and ω is the oscillation frequency.

The operator $A(h)$ is formally selfconjugate, which means that for any smooth vector functions z_1 and z_2 that vanish in the neighbourhood of the Γ boundary of region D the identity

$$(A(h)z_1, z_2) = (z_1, A(h)z_2) \tag{1.2}$$

holds.

Here and henceforth parentheses denote a scalar product in the three-component vector space of the functions

$$(z_1, z_2) = \iint\limits_D (u_1u_2 + v_1v_2 + w_1w_2)\,dx\,d\alpha$$

If the vector functions z_1 and z_2 do not become identically zero in the Γ neighbourhood, the boundary conditions which must be imposed on the components of the vector z_i to satisfy Eq. (1.2) are called selfconjugate. Henceforth we will consider only one of the so-called self conjugate boundary conditions /2/, namely those of hinged support (the Navier conditions)

$$(\omega)_\Gamma = \left(\frac{\partial^2 w}{\partial x^2}\right)_\Gamma = 0, \quad \left(\frac{\partial u}{\partial x} + \mu\frac{\partial v}{\partial\alpha}\right)_\Gamma = 0 \tag{1.3}$$
$$(v)_\Gamma = 0$$

The solution of the boundary value problem (1.1),(1.3) is understood in the weak sense. Let V be the set of vector functions $z(x, \alpha)$ with components from $C^\infty(\bar{D})$ that satisfy (1.3). We denote by $H_2(D)$ the Sobolev space of three-component vector functions which have square summable derivatives up to the second order inclusive. The closure of the set V in $H_2(D)$ is denoted by $W(D)$. The vector function $z \in W(D)$ is the solution of problem (1.1),(1.3) in the weak sense, if for any vector function $z_1 \in W(D)$ the equation

$$(A(h)z, z_1) = \lambda(hz, z_1)$$

holds.

Let us assume that the distribution of the circular cylindrical shell thicknesses is selected from the set of functions Q defined by the equation

$$Q = \left\{h(x, \alpha): \iint\limits_D \left[\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial\alpha}\right)^2\right] dx\,d\alpha \leqslant c^2, \quad 0 < a \leqslant h(x, \alpha) \leqslant b\right\} \tag{1.4}$$

where c, a, b are positive constants selected so that the set Q is non-empty.

The first of conditions (1.4) defines the limit on the growth of the derivatives of the admissible thickness distribution. The necessity for this condition follows from the results obtained in /3/ and guarantees the existence of a solution of the optimization problem which will be considered below.

From the mechanics point of view the absence of the first condition (1.4) means that the thickness gradient is arbitrary, which raises doubts about the validity of the hypothesis of the rectilinear normal element which is the basis of the theory of thin shell deformations.

The second of conditions (1.4) limits the magnitude of the minimum and maximum thickness distribution.

The compactness of the imbedding $H_2(D) \to L_2(D)$ implies that /4/ the spectral problem (1.1) has a sequence of non-zero solutions z^i_h $(i = 1, 2, 3, \ldots)$, that corresponds to the sequences of eigenvalues λ^i_h such that

$$(A(h)z_h^i, z) = \lambda^i_h(hz^i_h, z), \; \forall z \in W(D)$$
$$0 < \lambda_h^1 \leqslant \lambda_h^2 \leqslant \ldots \leqslant \lambda_h^i \leqslant \ldots$$

The subscript h on the functions z^i and the numbers λ^i is introduced to emphasise that the solution of problem (1.1) depends on the selected thickness distribution $h(x,\alpha)\in Q$.

2. **Statement of the optimization problem.** In many cases the natural oscillations of circular cylindrical shells can be separated into predominantly longitudinal, and transverse oscillations (for a fuller classification of oscillations see /2/). Each form of oscillation has its own eigenvalue for one and the same eigenfunction.

Consider the problem of designing a minimum volume (weight) shell for which the frequency of one of the predominant oscillation forms is not less than some specified fixed value. The design consists of making a suitable selection of the shell thickness distribution Q specified by Eq.(1.4).

Let λ° be a fixed positive number, and $\lambda(h)$ the smallest of the eigenvalues that correspond to predominantly transverse oscillations $h\in Q$. In this case the problem can be formulated as follows. Find an $h\in Q$ such that

$$\iint_D h(x,\alpha)\,dx\,d\alpha\to\min_h,\quad \lambda(h)\geqslant\lambda^\circ$$

Obviously the solution of problem (2.1) does not exist for all λ°. As a reasonable λ° we can take the smallest of the eigenvalues that corresponds to predominantly transverse oscillations for a shell of constant thickness h_0, $a\leqslant h_0\leqslant b$. Problem (2.1) then becomes a problem of designing an optimal shell of variable thickness of minimum volume (weight) whose eigenvalue, which corresponds to one of the chosen form of predominant oscillations, is not less than the respective eigenvalue of a shell of constant thickness.

3. **The asymptotic approach to the problem of optimization.** The solution of the problem is linked to the need for a multiple variation of h from Q and to finding a solution of (1.1),(1.3), which gives rise to well-known difficulties. On the other hand, in many important cases the range variation of the thickness distribution is very narrow, which enables us to use asymptotic methods by considering the shell as a weakly controllable system.

Let us assume that the function $h(x,\alpha)\in Q$ varies as follows:

$$h(x,\alpha)=h_0+\varepsilon h_1(x,\alpha);\ \varepsilon>0,\ h_0=\text{const}\tag{3.1}$$

The set Q defined by Eq.(1.6) then becomes the set

$$Q_1=\left\{h_1(x,\alpha):\iint_D\left[\left(\frac{\partial h_1}{\partial x}\right)^2+\left(\frac{\partial h_1}{\partial\alpha}\right)^2\right]dx\,d\alpha\leqslant c^2,\quad |h_1(x,\alpha)|\leqslant 1\right\}\tag{3.2}$$

Without loss of generality, we can assume that $h_0=1$, which can always be achieved by introducing the new dimensionless function

$$h'(x,\alpha)=\frac{h(x,\alpha)}{h_0}=1+\varepsilon\frac{h_1(x,\alpha)}{h_0}=1+\varepsilon h_1'(x,\alpha)\tag{3.3}$$

Hence the limit on the minimum and maximum value of $h(x,\alpha)$ in (1.4) i.e. $0<a\leqslant h(x,\alpha)\leqslant b$ becomes the limit $|h_1'(x,\alpha)|\leqslant 1$, if $a=h_0-\varepsilon$ and $b=h_0+\varepsilon$, i.e. when the maximum range of variation of the thickness h' is equal to 2ε.

In deriving the equations of the theory of shells it is assumed that quantities of the order of $(h/R)^2$ can be neglected. Allowing for this, we have

$$(h_0/R)^2\ll\varepsilon h_1/R\sim\varepsilon\, h_0/R\ll h_0/R$$

from which follows that $h_0/R\ll\varepsilon\ll 1$, otherwise the quantity $\varepsilon\, h_1/R$ becomes comparable with the error of the mathematical model of the problem.

Henceforth the primes on the functions h and h_1 are omitted for convenience.

According to Rellich's theorem /5-7/ the spectrum of the operator $A(h)$ defined by Eqs. (1.1) for $h=1+\varepsilon h_1$ can be represented in the form of an analytic perturbation of the spectrum of the operator $A(h_0)=A^\circ$. The operator A° is defined by formulas (1.1), if we set $h(x,\alpha)=1$. Using (3.1) we represent the operator $A(h)$ in the form of a sum in powers of the parameter ε

$$A(h)=\sum_{i=0}^{3}\varepsilon^i A^i(h_1)\tag{3.4}$$

To determine the components of the operator $A^1(h_1)$ it is necessary to substitute in formulas (1.1) $h_1(x,\alpha)$, for $h(x,\alpha)$ and $3h_1(x,\alpha)$, for $h^3(x,\alpha)$, while the coefficient δ_0^2 (taking into account the substitution (3.3)) becomes equal to $\delta_1^2=h_0^2(12R^2)^{-1}$.

The eigenvalues of the spectral problem and the eigenfunctions may be represented in the form of series in powers of ε

$$\lambda^k(h)=\sum_{i=0}^{\infty}\varepsilon^i\lambda^{k,i}(h_1),\quad z_h{}^k=\sum_{i=0}^{\infty}\varepsilon^i z^{k,ih_1}. \tag{3.5}$$

Substituting (3.4) into (3.5) and collecting terms of the zeroth and first power of ε, we obtain the boundary value problem for the zeroth and first approximation of the input spectral problem (1.1) of the form

$$(A^\circ z^{k,0},\ z)=\lambda^{k,0}(z^{k,0},z),\quad \forall z\in W\,(D) \tag{3.6}$$

$$(A^\circ z^{k,1},\ z)+(A^1(h_1)\,z^{k,0},\ z)=\lambda^{k,0}\,(z^{k,1},\ z)+\lambda^{k,0}\,(h_1 z^{k,0},\ z)\ +\lambda^{k,1}\,(h_1)\,(z^{k,0},\ z),\ \forall z\in W\,(D) \tag{3.7}$$

Setting $z=z^{k,0}$ in (3.6) and (3.7) and using the self conjugacy of the operator A°, we obtain the formula

$$\lambda^{k,1}\,(h_1)=(A^1\,(h_1)\,z^{k,0},\ z^{k,0})-\lambda^{k,0}\,(h_1 z^{k,0},\ z^{k,0}) \tag{3.8}$$

$$(z^{k,0},\ z^{k,0})=1$$

which determines the first correction to the calculation of the eigenvalue $\lambda^k(h)$ in the form of a linear functional of $h_1\in Q_1$.

Taking the above considerations into account, we revert to the input problem (2.1). By virtue of

$$\iint\limits_D(1+\varepsilon h_1)\,dxd\alpha=S\ +\varepsilon\iint\limits_D h_1 dxd\alpha$$

where S is the shell surface area, the input functional takes the form

$$\iint\limits_D h_1 dxd\alpha\to\min_{h_1},\quad h_1\in Q_1 \tag{3.9}$$

Selecting $\lambda^\circ=\lambda^{k,0}$ in (2.1) and using expansion (3.5) and the arbitrariness of ε, we obtain the limit of the magnitude of the first correction of the eigenvalue $\lambda^k(h)$ in the form

$$\lambda^{k,1}\,(h_1)\geqslant 0,\ \ h_1\in Q_1 \tag{3.10}$$

where the quantity $\lambda^{k,1}\,(h_1)$ is defined by Eq.(3.8). Finally we obtain the problem

$$\iint\limits_D h_1 dxd\alpha\to\min_{h_1},\quad \lambda^{k,1}(h_1)\geqslant 0,\quad h_1\in Q_1 \tag{3.11}$$

The set of functions h_1 · is such that $h_1\in Q_1$ and $\lambda^{k,1}\,(h_1)\geqslant 0$ are convex and closed in the topology of Sobolev space, and are summable with the square of the functions together with their derivatives up to the first order. Hence the linear functional (3.9) reaches its maximum on the set indicated, and any local extremum is also global. It can be shown /8/ that in conformity with functional (3.9) the solution of problem (3.11) differs from the optimal in the class of distributions Q by a quantity of the order of ε^2, when $b-a=2\varepsilon$.

The solution of problem (3.11) is made easier by the fact that to calculate the linear functional $\lambda^{k,1}\,(h_1)$ it is sufficient to know only the eigenfunctions and eigenvalues of the unperturbed problem.

4. **Axisymmetric oscillations.** Consider the axisymmetric oscillations of a freely supported circular cylindrical shell. The form of oscillations is in this case determined by the vector function of a single variable $x, x\in[0,k], k=l/R$. The equations of state (1.1) decompose into the boundary value problem for the component that determines the frequency of torsional oscillations

$$-\frac{(1-\mu)}{2}\frac{d}{dx}\left(h\frac{dv}{dx}\right)-2\delta_1{}^2(1-\mu)\frac{d}{dx}\left(h^3\frac{dv}{dx}\right)=\lambda hv \tag{4.1}$$

$$v\,(0)=v\,(k)=0 \tag{4.2}$$

and the boundary value problem for the combined longitudinal and transverse oscillations

$$-\frac{d}{dx}\left(h\frac{du}{dx}\right)+\mu\frac{d}{dx}(hw)=\lambda hu \tag{4.3}$$

$$-\mu h\frac{du}{dx}+hw+\delta_1{}^2\frac{d^2}{dx^2}\left(h^3\frac{d^2w}{dx^2}\right)=\lambda hw$$

$$w\,(0)=w\,(k)=0,\quad \frac{d^2w}{dx^2}(0)=\frac{d^2w\,(k)}{dx^2}=0 \tag{4.4}$$

$$\frac{du}{dx}(0)=\frac{du\,(k)}{dx}=0,\quad \delta_1{}^2=\frac{h_0{}^2}{12R^2}$$

Let the shell thickness $h\,(x)$ vary as given by (3.1). The set Q_1 in this case consists of functions such that

$$\int_0^k \left(\frac{dh_1}{dx}\right)^2 dx \leqslant c^2, \quad |h_1| \leqslant 1 \tag{4.5}$$

For the zeroth approximation we have the equation

$$-\frac{(1-\mu)}{2}\frac{d^2v^\circ}{dx^2} - 2(1-\mu)\,\delta_1^2\frac{d^2v^\circ}{dx^2} = -\lambda^\circ v^\circ \tag{4.6}$$

with boundary conditions (4.2) for the function $v^\circ(x)$ and the set of equations

$$-\frac{d^2u^\circ}{dx^2} + \mu\frac{dw^\circ}{dx} = \lambda^\circ u^\circ \tag{4.7}$$
$$-\mu\frac{du^\circ}{dx} + w^\circ + \delta_1^2\frac{d^4w^\circ}{dx^4} = \lambda^\circ w^\circ$$

with boundary conditions (4.4) for the functions $u^\circ(x)$ and $w^\circ(x)$. The eigenvalues of problems (4.6) and (4.7) can be obtained in this case by taking the natural oscillations in the form

$$u^\circ(x) = u_0\cos px, \quad v^\circ(x) = v_0\sin px, \quad w^\circ(x) = w_0\sin px \tag{4.8}$$

where u_0, w_0, v_0 are certain constants and $p = \pi n/k$, $n = 1, 2, \ldots$

The frequencies corresponding to torsional oscillations defined by Eq.(4.6) are specified by the formula

$$\lambda_1^\circ = \frac{1-\mu}{2}p^2(1+4\delta_1^2) \tag{4.9}$$

For combined transverse and longitudinal oscillations we have

$$\lambda_{2,3}^\circ = \frac{1}{2}[1 + p^2 + \delta_1^2p^4 \pm (1 - p^2 + \delta_1^2p^4 + 4\mu^2p^2)^{1/2}] \tag{4.10}$$

Let us formulate the problem of choosing the distribution of the thicknesses in the form (3.1) that satisfy the limits (4.5), in order to minimize the volume (weight) of a cylindrical shell whose eigenvalue corresponds to predominantly transverse osillations that are not less than the same eigenvalue of a shell of constant thickness h_0. To separate the first natural frequency, corresponding to predominantly transverse oscillations we will investigate the ratio of the amplitudes of the combined longitudinal and transverse oscillations of the form (4.8) by substituting them into (4.7). This yields the ratio

$$\varkappa = \frac{u_0}{w_0} = \frac{\mu p}{\lambda_i^\circ - p^2}, \quad i = 2,3 \tag{4.11}$$

From (4.11) we can establish for the eigenvalue λ_3° in (4.10) that for $p > 1 - \delta_1^2 = a_0^2$ we have $|\varkappa| < 1$, and for $p < a_0^2$, we have $|\varkappa| > 1$. The values of p are calculated to within quantities of the order of δ_1^4. Similarly $|\varkappa| > 1$ when $p > a_0^2$, and $|\varkappa| < 1$ when $p < a_0^2$ for the eigenvalue λ_2° in (4.10). This result was obtained earlier in /9/ for the special case of a cylindrical shell.

Using formula (3.8) we can find the correction to the values of the frequencies λ_2° and λ_3°, when the shell thickness varies as (3.1).

The operator $A^1(h_1)$ is represented by the components

$$A_{11}^1 = -\frac{d}{dx}\left(h_1\frac{d}{dx}\right), \quad A_{12}^1 = \mu\frac{d}{dx}h_1$$
$$A_{21}^1 = -\mu h_1\frac{d}{dx}, \quad A_{22}^1 = h_1 + 3\delta_1^2\frac{d^2}{dx^2}\left(h_1\frac{d^2}{dx^2}\right)$$

From (3.8) we have

$$\lambda_i^{\circ,1}(h_1) = \frac{2r_1}{k}\int_0^k h_1(x)\sin^2 px\,dx, \quad i = 2,3$$
$$r_1 = [p^2u_0^2 + 2\mu pu_0w_0 + w_0^2 + 3p^4\delta_1^2w_0^2 - \lambda_i^\circ(u_0^2 + w_0^2)]\,e^{-1}$$
$$e = u_0^2 + w_0^2$$

From formulas (4.10) and (4.11), after transformations, we obtain

$$\lambda_i^{\circ,1}(h_1) = r_2\int_0^k h_1(x)\sin^2 px\,dx, \quad i = 2,3 \tag{4.12}$$
$$r_2 = \frac{4\delta_1^2p^4(\lambda_i^\circ - p^2)^2}{k[(\lambda_i^\circ - p^2)^2 + \mu^2p^2]} = \frac{4\delta_1^2p^4}{k(1+\varkappa^2)} > 0$$

The expression for the first correction to the eigenvalue λ_1° for torsional oscillations has a similar form

$$\lambda_1^{\circ,1}(h_1) = r_3\int_0^k h_1\sin^2 px\,dx, \quad r_3 = 8k^{-1}p^2\delta_1^2(1-\mu) > 0 \tag{4.13}$$

Since the linear functionals $\lambda_i^{0,1}(h_1)$ $(i = 2, 3)$ and $\lambda_1^{0,1}(h_1)$ differ only by constant positive multipliers, the input problem can be extended to all three forms of oscillations. Let us determine the thickness distribution of the form (3.1) with limits (4.5) such that the shell weight is a minimum, and the respective frequencies of longitudinal, transverse, and also torsional oscillations are not less than the respective frequencies of a shell of constant thickness h_0. This implies the need to solve the following problem:

$$\int_0^k h_1 dx \to \min_{h_1}, \quad h_1 \in Q_1, \quad \int_0^k h_1 \sin^2 px dx \geqslant 0 \tag{4.14}$$

The solution of variational problem (4.14) can be obtained in analytic form /10,11/. The optimum thickness distribution obtained by exact solution of the problem $c^2 = 16$ is shown in

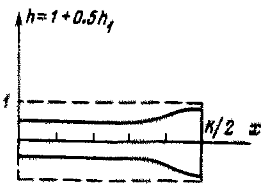

Fig.1

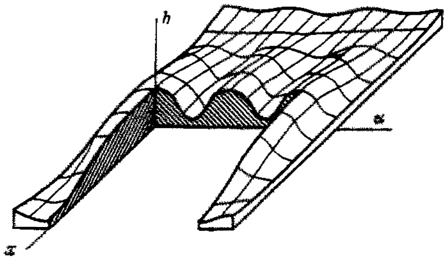

Fig.2

Fig.1 for minimal $p = \pi/k$. The solution has a particularly simple form *) when $c^2 < (8p^2k)/9$

$$h_1(x) = -\frac{(1 + 2\cos px)}{3}, \quad x \in [0, k]$$

The relative gain in weight in this case is $33\varepsilon\%$. For the case represented in Fig.2 the gain is close to $50\varepsilon\%$.

An analysis of the results shows that for all three forms of axisymmetric oscillations of a cylindrical shell the optimum thickness distribution in the middle of the shell is a thickening in the form of a rib which divides the shell into two shorter sections of smaller thickness. A similar result was obtained in /12/ by numerical calculations for shells of reasonable length.

5. The non-axisymmetric case.

Consider the set of equations defining the natural oscillations of a cylindrical shell (1.1) with boundary conditions that correspond to resting on a hinged joint (1.3). The form of the natural oscillations of the shell of constant thickness is then

$$\begin{aligned} u(x, \alpha) &= u_0 \cos px \sin q\alpha, \quad v(x, \alpha) = v_0 \sin px \cos q\alpha \\ w(x, \alpha) &= w_0 \sin px \sin q\alpha, \quad p = \pi n/k, \quad q = m, \quad m, n = 1, 2, \ldots \end{aligned} \tag{5.1}$$

where u_0, v_0, w_0 are constants that are determined, apart from some constant multiplier, in the solution of the linear system of equations obtained by the substitution of (5.1) into the equation of state (1.1). The eigenvalues λ_i^k $(i = 1, 2, 3, k = 0, 1, 2, \ldots)$ are obtained from the condition for the determinant of the set to be zero.

Let us formulate the problem of designing a shell of minimum volume (weight) whose first frequencies, which correspond to quasi transverse oscillations for which $w_0 > \max\{u_0, v_0\}$, and quasitangential oscillations for which $w_0 < \max\{u_0, v_0\}$ /2/, are not less than the respective frequencies of a shell of constant thickness h_0.

Applying the asymptotic method of Sect. 3, we obtain, using formulas (3.8), the following expression for the first correction to the calculation of the eigenvalue λ_i^0:

$$\lambda_i^{0,1}(h_1) = \frac{2}{3k\pi} \int_0^k \int_0^{2\pi} [A \sin^2 px \sin^2 q\alpha + B \cos^2 px \cos^2 q\alpha] h_1(x, \alpha) dx d\alpha \tag{5.2}$$

$$A = \{u_1^2 p^2 + 2\mu u_1 p (v_1 q + 1) + (v_1 q + 1)^2 + 3\delta_1^2 [(v_1 + q)^2 q^2 + 2\mu p^2 q (v_1 + q) + p^4]\} (u_1^2 + v_1^2 + 1)^{-1} - \lambda_i^0$$

$$B = \frac{1 - \mu}{2} \frac{(q u_1 + p v_1)^2 + 12\delta_1^2 p^2 (v_1 + q)^2}{u_1^2 + v_1^2 + 1}$$

*) Bratus' A.S. and Kartvelishvili V.M., The method of perturbations in problems of optimization of the stability, oscillation frequencies, and strength of elastic plates of variable thickness. Preprint No. 180, Inst. Of Problems of Mechanics, AN SSSR. Moscow, 1981.

$$u_1 = \frac{u_0}{w_0}, \quad v_1 = \frac{v_0}{w_0}, \quad \delta_1^2 = \frac{h_0^2}{12R^2}, \quad i = 1, 2, 3$$

The values of λ_i° and q depend on the values of k, δ_1^2 for minimum $p = \pi/k$.

Consider the case when $h_0/R = 0.01$. Direct calculations using tables /13/ show that the quantity A defined by Eq.(5.2) remains positive for $k = 1, 2, \ldots, 9$.

This means that the variational problem has the form

$$\int_0^k \int_0^{2\pi} \left[A \sin^2 \frac{\pi x}{k} \sin^2 q\alpha + B \cos^2 \frac{\pi x}{k} \cos^2 q\alpha \right] h_1(x, \alpha)\, dx\, d\alpha \geqslant 0 \tag{5.3}$$

$$\int_0^k \int_0^{2\pi} h_1(x, \alpha)\, dx d\alpha \to \min_{h_1}, \quad h_1 \in Q_1$$

We select as A in (5.3) the minimal positive value of all magnitudes defined in (5.2) for various λ_i°, $i = 1, 2, 3$.

In the cases considered here minimum A corresponds to the eigenvalue λ_3° which relates to quasitransverse oscillations of the shell.

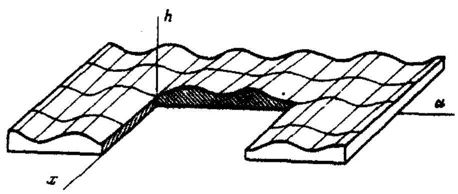

Fig.3

The solution of problem (5.3) can be obtained in some cases in analytic form /10, 11/. In particular, the solution that does not reach the limit $|h_1| = 1$, everywhere, except on the set of zero measure (lines and points), has the form

$$h_1 = \left[\cos \frac{2\pi}{k} x \cos 2q\alpha - 2d \left(\cos \frac{2\pi}{k} x + \cos 2q\alpha \right) - 2d^2 - \frac{1}{4} \right] \left[4d^2 - 2d + \frac{3}{4} \right]^{-1}, \quad d = \frac{A - B}{A + B}$$

By the previous remark, the parameter ε must satisfy the condition $\varepsilon \gg 0.01$. In this case the integral constraint in (3.2) was taken in the form

$$\int_0^k \int_0^{2\pi} \left[\frac{4\pi^2}{k^2} \left(\frac{\partial h_1}{\partial x} \right)^2 + \frac{1}{4q^2} \left(\frac{\partial h_1}{\partial \alpha} \right)^2 \right] dx d\alpha \leqslant c^2, \quad c = \text{const} \tag{5.4}$$

The constraint (5.4) ensures a uniform growth of the derivatives along the directions that are parallel to the shell directrices and generatrices. The relative gain with respect to the functional is $\eta = (2d^2 + 0.25)(4d^2 - 2d + 0.75)^{-1}$. For example, in the case when $\mu = 0.3$, $h_0/R = 0.01$ and $k = 4$ we have $\eta = 55\varepsilon\%$; when $k = 6$ $\eta = 33\varepsilon\%$.

When the constraint $|h_1| = 1$ is taken into account, the solution of problem (5.3) can be obtained numerically /11/. The relative gains are then approximately 1.5 times greater. If only positive values of quantities A are considered, then $|d| < 1$. The maximum gain in this range of variation of d is $100\varepsilon\%$ when $d = 0.5$, and the minimum gain is $10\varepsilon\%$ when $d = -0.125$.

The thickness distribution shown in Fig.2 corresponds to $k = 6$, $h_0/R = 0.01$, $\mu = 0.3$. The shell is presented in developed form, and for better visualization its middle section is partly cut out. The form of the thickness distribution obtained is a surface with symmetrically located points of local mixima and minima which are staggered, and alternately turn to the centre of curvature and away from it. A similar form is obtained for other values. Note the lines on the shell development which connect the local points of maximum thickness. This is the line that passes through the middle of the shell and is parallel to the directrix, and the lines inclined to the shell directrix at an angle $\gamma = \operatorname{arctg}(kq/2\pi)$. The disposition of these lines provides additional information that can be used to select the optimum directions of strengthening elements.

Thickness distributions are shown in Fig.3 on the assumption that h_1 is a function of only one variable α when $h_0/R = 0.01$, $\mu = 0.3$ and $k = 4$. The relative gain with respect to the functional reaches $40\varepsilon\%$. In this case the shell has a number of bulges and troughs which form a corrugated surface. The number of bulges is determined by q for minimum $p = \pi/k$.

We note in conclusion that the proposed asymptotic method enables the sensitivity of the shell natural frequency and of the thickness distribution near the specified supporting solution to be investigated. It can also be applied to other types of shells in problems of stability and oscillation frequency optimization, as well as in problems of shell bending under distributed loads.

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SOME PROBLEMS OF THE STABILITY OF CYLINDRICAL AND CONICAL SHELLS*

P.E. TOVSTIK

The problem of the buckling of a membrane state of stress of a thin elastic shell is considered in a linear approximation. It is assumed that the buckling is accompanied by the formulation of a large number of dents. In the simplest case when the initial stresses and curvature of the middle surface are constant, the dents cover the whole shell surface /1–3/. If the quantities mentioned are not constant, the buckling pattern is complicated; localization of the dents can occur in the neighbourhoods of certain "weakest" lines /3–5/ or points /6/. The problem of the buckling of a shell of zero curvature is considered below. This is characterized, by the fact that the dents are stretched strongly along asymptotic lines and are localized near one (the weakest). The method is applicable to convex conical and cylindrical shells of medium length and not absolutely circular section; the shell edges are not necessarily plane curves. The two-dimensional problem reduces to a sequence of one-dimensional boundary value problems, while for a cylindrical shell, under certain particular assumptions, the approximate solution is obtained in closed form. A conical shell is considered, and the changes which must be made in the case of a cylindrical shell are outlined.

1. Let us introduce an orthogonal system of coordinates s, φ on the middle surface of a conical shell, where $s = s^\circ R^{-1}, s^\circ$ is the distance to the apex of the cone, R is the characteristic dimension of the middle surface, and φ is a coordinate on the directrix, selected in such a manner that the first quadratic form of the surface has the form $d\sigma^2 = R^2 (ds^2 + s^2 d\varphi^2)$. Here the radius of curvature is $R_2 = Rsk^{-1}$. Let the shell be closed in the φ direction and bounded by two edges (φ_1 is the length of the curve formed when the cone and a sphere of radius R with centre at the apex of the cone intersect)

$$s_1(\varphi) \leqslant s \leqslant s_2(\varphi),\ 0 \leqslant \varphi \leqslant \varphi_1 \tag{1.1}$$

We will use the set of shallow-shell equations

$$\varepsilon^4 \Delta^2 w + \lambda \Delta_T w + \Delta_k \Phi = 0,\ \ \varepsilon^4 \Delta^2 \Phi - \Delta_k w = 0 \tag{1.2}$$

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